Additional Problems

Problem 1.21 Determine the direction of propagation of the following harmonic traveling waves:

a) $\Psi(z,t) = A\sin(kz - \omega t)$
b) $\Psi(y,t) = A\cos(\omega t - ky)$
c) $\Psi(x,t) = A\cos(\omega t + kx)$
d) $\Psi(x,t) = A\cos(-\omega t - kx)$

Problem 1.22 Show that the *Gaussian* wave $\Psi(x,t) = Ae^{-a(bx-ct)^2}$ is a solution to the one-dimensional wave equation. If $a = 5.00$, $b = 10.0$, $c = 100$, with $a(bx - ct)^2$ unitless, determine the wave speed.

Problem 1.23 Sketch or plot the following wavefunction at times $t = 0$, $t = 0.5\,s$, and $t = 1.0\,s$:

$$\Psi(x,t) = \frac{1.0}{1 + (x + 10t)^2}$$

Problem 1.24 A 1-D harmonic traveling wave that travels in the $-y$ direction has amplitude 10 (unitless), wavelength $10.0\,m$, period $2.0\,s$ and initial phase π. Using the complex representation, find an expression for this wave that uses angular frequency and propagation constant.

Problem 1.25 Light from a helium-neon laser has a wavelength of $633\,nm$ and a wave speed of $3.00 \times 10^8\,m/s$. Find the frequency, period, angular frequency, and wave number for this light.

Problem 1.26 Consider a harmonic wave given by

$$\Psi(x,t) = U(x,y,z)e^{-i\omega t}$$

where $U(x,y,z)$ is called the *complex amplitude*. Show that U satisfies the *Helmholz equation*:

$$\left(\nabla^2 + k^2\right) U(x,y,z) = 0$$

where k is the propagation constant.

Problem 1.27 Show that a complex number divided by its complex conjugate gives a result whose magnitude is one.

Problem 1.28 Show that $z = \frac{\sqrt{2}}{2}(1 + i)$ is a square root of i. Find another one.

Problem 1.29 Find the real and imaginary parts of

a) $z = \left(2e^{i\frac{\pi}{4}}\right)^3$
b) $z = (2 + 3i)^3$

Problem 1.30 Hyperbolic sines and cosines are defined as follows: $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$. Show that $\sin(ix) = i\sinh(x)$ and $\cos(ix) = \cosh(x)$.

Problem 1.31 Find the Taylor series expansions for hyperbolic sine and hyperbolic cosine.

Problem 1.32 Consider a vector $\vec{v}$, and let a be the angle between $\vec{v}$ and the positive x-axis, b be the angle between $\vec{v}$ and the positive y-axis, and c be the angle between $\vec{v}$ and the positive z-axis. Define the *direction cosines* α, β, and γ as follows:

$$\alpha = \cos a = \frac{v_x}{|v|}$$

$$\beta = \cos b = \frac{v_y}{|v|}$$

$$\gamma = \cos c = \frac{v_z}{|v|}$$

a) Show that $\alpha^2 + \beta^2 + \gamma^2 = 1$.

b) Show that the function

$$\Psi(x, y, z, t) = Ae^{i[k(\alpha x + \beta y + \gamma z) \mp \omega t]}$$

is a three-dimensional plane wave that solves the differential wave equation in Cartesian coordinates.

Problem 1.33 The Laplacian in cylindrical coordinates (ρ, φ, z) is given by

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

A *cylindrical wave* has a wavefront that is constant on a cylinder. In other words, it does not depend upon φ or z. Show that such a wave has an amplitude that is inversely proportional to $\sqrt{z}$.

Problem 1.34 Show that the spherically symmetric wave equation can be written as

$$\frac{\partial^2}{\partial r^2}[r\Psi(r, t)] = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}[r\Psi(r, t)]$$

which is a *linear* wave equation in the quantity $r\Psi(r, t)$. Show that Equations 1.59 and 1.60 are solutions.

Problem 1.35 *Photons* are particles of light with energy and momentum given by

$$E = hf$$

$$p = \frac{h}{\lambda}$$

where f is the light frequency, λ is the light wavelength, and h is *Planck's constant*: $h = 6.626 \times 10^{-34}\ J \cdot s$. Show that for photons, $E = cp$.

Additional Problems

Problem 2.18 An *elliptically polarized* electromagnetic wave has perpendicular components of $\vec{E}$ that are *out of phase*. For example,

$$E_x = E_{0x} e^{i(kz - \omega t)}$$

$$E_y = E_{0y} e^{i(kz - \omega t + \varphi)}$$

with E_{0x} and E_{0y} real. Find the magnetic field components of this wave, assuming that it travels in vacuum.

Problem 2.19 Show explicitly that $\vec{B}(x, y, z, t) = \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r} \mp \omega t + \varphi)}$ is a solution to the differential wave equation for $\vec{B}$.

Problem 2.20 For an electromagnetic wave, show that $\vec{k} \cdot \vec{B} = 0$, and thus that $\vec{B}$ is perpendicular to the direction of propagation.

Problem 2.21 For an electromagnetic wave, show that $\vec{k} \times \vec{E} = \omega \vec{B}$.

Problem 2.22 For an electromagnetic wave, show that $-\vec{k} \times \vec{B} = \epsilon \, \mu \, \omega \, \vec{E}$.

Problem 2.23 Let $f(x, t) = f_0 \, e^{i(kx - \omega t)}$ with f_0 a real constant.

a) Find $\mathrm{Re}[f^2]$.

b) Find $(\mathrm{Re}[f])^2$.

c) Show that $(\mathrm{Re}[f])^2 = \frac{1}{2}\left(f_0^2 + \mathrm{Re}[f^2]\right)$.

Problem 2.24 Show that the average of $\sin\left(\vec{k}\cdot\vec{r} - \omega t + \varphi\right)$ and $\cos\left(\vec{k}\cdot\vec{r} - \omega t + \varphi\right)$ over many cycles is zero.

Problem 2.25 Show that the average of

$$\sin\left(\vec{k}\cdot\vec{r} - \omega t + \varphi\right)\cos\left(\vec{k}\cdot\vec{r} - \omega t + \varphi\right)$$

over many cycles is zero.

Problem 2.26 Show that an electromagnetic wave with spherical wavefronts of radius r has an irradiance that varies as $1/r^2$.

Problem 2.27 Calculate the peak electric and magnetic fields $1.0\,km$ from a $1.0\,MW$ radio station, assuming that it radiates electromagnetic waves as an isotropic point source.

Problem 2.28 It is often convenient to define the *optical thickness* of a material as nd where n is the index of refraction and d is the physical thickness.

a) Find the optical thickness for vacuum and glass ($n = 1.5$) for a physical thickness of one meter.

b) Find the time it takes light to travel $d = 1.0\,m$ in vacuum and in glass of index 1.50.

Problem 2.29 The maximum electric field sustainable in a material before electrical breakdown is called the *dielectric strength*. For dry air at STP, the dielectric strength is about $3.0 \times 10^6\ V/m$.

a) Use this to estimate the maximum irradiance of a laser beam that can propagate through air.

b) If the beam profile is uniform and the beam diameter is $10\,cm$, what is the maximum beam power if it is to travel through air?

c) What radiation pressure would this beam exert on absorbing and reflecting surfaces? In each case, what net force would be exerted by a $10\,cm$-diameter beam?

Problem 2.30 A certain pulsed laser has a maximum power output of $10^7\ W$.

a) Find the maximum values of $\vec{E}$ and $\vec{B}$ if the beam diameter is $10\,cm$. Assume a uniform beam profile.

b) Find the maximum values of $\vec{E}$ and $\vec{B}$ if the beam is focused to a spot of diameter $100\ \mu m$.

Problem 2.31 Calculate the electric and magnetic fields at the top of Earth's atmosphere where the solar irradiance is $1340\ W/m^2$.

Problem 2.32 Find the frequency of a photon whose momentum is that of a $0.300\ g$ BB traveling at $100\,m/s$.

Problem 2.33 Determine the responsivity at $550\,nm$ of a photomultiplier tube that has a gain of 10^6 and a quantum efficiency of 30%. Repeat for a phototube with unity gain and the same quantum efficiency.

Problem 2.34 A $100\ W$ laser beam with $2.00\,mm$ beam diameter illuminates a surface that absorbs 40% and reflects 60%. Find the net force on the surface due to radiation pressure.

Problem 2.35 What is the minimum detectable wavelength of a photodetector with a cathode workfunction of $2.26\,eV$?

Problem 2.36 Under ideal circumstances, the human eye can detect a photon flux of about 250 $photons/s$ of $550\,nm$ light. If the eye has a pupil diameter of $8.0\,mm$, what irradiance does this correspond to?