Reflection and Transmission Coefficients

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- $r_{12}$ – reflection in $n_1$ off $n_2$
- $t_{12}$ – transmission from $n_1$ into $n_2$
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For a slab of thickness $\Delta$ on a substrate, the transmission and reflection coefficients at each interface are labeled:

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Reflection and Transmission Coefficients

For a slab of thickness $\Delta$ on a substrate, the transmission and reflection coefficients at each interface are labeled:

\[ n_0 \bullet r_{10} \bullet t_{01} \bullet n_1 \]

- $r_{01}$ – reflection in $n_0$ off $n_1$
- $t_{01}$ – transmission from $n_0$ into $n_1$

\[ n_1 \bullet r_{12} \bullet t_{12} \bullet n_2 \]

- $r_{12}$ – reflection in $n_1$ off $n_2$
- $t_{12}$ – transmission from $n_1$ into $n_2$

\[ n_1 \bullet r_{10} \bullet t_{10} \bullet n_0 \]

- $r_{10}$ – reflection in $n_1$ off $n_0$
- $t_{10}$ – transmission from $n_1$ into $n_0$

Build the composite reflection coefficient from all possible events
Overall Reflection from a Slab

The composite reflection coefficient for each ray emerging from the top surface is computed
Overall Reflection from a Slab

The composite reflection coefficient for each ray emerging from the top surface is computed:

\[ n_0 \rightarrow r_{01} \rightarrow n_1 \rightarrow r_{12} \rightarrow n_2 \]

Inside the medium, the x-rays are travelling an additional \( 2\Delta \) per traversal, which creates a phase shift of:

\[ \Phi_2 = e^{i2(\kappa_1 \sin \alpha_1)\Delta} = e^{iQ_1 \Delta} \]

which multiplies the reflection coefficient at each pass through the slab.
Overall Reflection from a Slab

The composite reflection coefficient for each ray emerging from the top surface is computed:

\[ r_{01} + t_{01} r_{12} t_{10} \]

Inside the medium, the x-rays are travelling an additional 2\( \Delta \) per traversal, which creates a phase shift of:

\[ p_2 = e^{i2(k_1 \sin \alpha_1 \Delta)} = e^{iQ_1 \Delta} \]

which multiplies the reflection coefficient at each pass through the slab.
Overall Reflection from a Slab

The composite reflection coefficient for each ray emerging from the top surface is computed

\[ r_{01} \]

\[ + \]

\[ t_{01} r_{12} t_{10} \]

\[ + \]

\[ t_{01} r_{12} r_{10} r_{12} t_{10} \]
Overall Reflection from a Slab

The composite reflection coefficient for each ray emerging from the top surface is computed

\[
\begin{align*}
    r_{01} & \quad + \\
    t_{01}r_{12}t_{10} & \quad + \\
    t_{01}r_{12}r_{10}r_{12}t_{10}
\end{align*}
\]

Inside the medium, the x-rays are travelling an additional \(2\Delta\) per traversal, which creates a phase shift of

\[p^2 = e^{i2(k_1 \sin \alpha_1)\Delta}\]
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Inside the medium, the x-rays are travelling an additional \(2\Delta\) per traversal, which creates a phase shift of

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\[ \Delta \]
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The composite reflection coefficient for each ray emerging from the top surface is computed

\[ n_0 \]
\[ n_1 \]
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Overall Reflection from a Slab

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\[ r_{01} + t_{01}r_{12}t_{10} \cdot p^2 + t_{01}r_{12}r_{10}r_{12}t_{10} \cdot p^4 \]

Inside the medium, the x-rays are travelling an additional $2\Delta$ per traversal, which creates a phase shift of

\[ p^2 = e^{i2(k_1 \sin \alpha_1)\Delta} = e^{iQ_1\Delta} \]

which multiplies the reflection coefficient at each pass through the slab.
Composite Reflection Coefficient

The composite reflection coefficient can now be expressed as a sum:

\[
\begin{align*}
    r_{\text{slab}} &= r_{01} + t_{01} r_{12} t_{10} + t_{01} r_{10} r_{212} t_{10} + t_{01} r_{210} r_{312} t_{10} + \cdots \\
&= r_{01} + t_{01} t_{10} r_{12} p_2^\infty \sum_{m=0}^{\infty} (r_{10} r_{12} p_2)^m
\end{align*}
\]

Factoring out the second term from all the rest and summing the geometric series as previously.

The individual reflection and transmission coefficients can be determined using the Fresnel equation. Recall:

\[
\begin{align*}
    r &= \frac{Q - Q'}{Q + Q'}, \\
    t &= \frac{2Q}{Q + Q'}
\end{align*}
\]
The composite reflection coefficient can now be expressed as a sum

\[ r_{slab} = r_{01} + t_{01} r_{12} t_{10} p^2 + t_{01} r_{10} r_{12}^2 t_{10} p^4 + t_{01} r_{10}^2 r_{12}^3 t_{10} p^6 + \cdots \]
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\[ r_{\text{slab}} = r_{01} + t_{01} t_{10} r_{12} p^2 \sum_{m=0}^{\infty} (r_{10} r_{12} p^2)^m \]

\[ = r_{01} + t_{01} t_{10} r_{12} p^2 \frac{1}{1 - r_{10} r_{12} p^2} \]

The individual reflection and transmission coefficients can be determined using the Fresnel equation. Recall

\[ r = y - y' \]
\[ t = 2 (y y' + y'y) \]
The composite reflection coefficient can now be expressed as a sum

\[ r_{slab} = r_{01} + t_{01} r_{12} t_{10} p^2 + t_{01} r_{10} r_{12}^2 t_{10} p^4 + t_{01} r_{10}^2 r_{12}^3 t_{10} p^6 + \cdots \]

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\[ r = \frac{Q - Q'}{Q + Q'} \]
The composite reflection coefficient can now be expressed as a sum

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Factoring out second term from all the rest

\[ r_{slab} = r_0 + t_0 t_{10} r_{12} p^2 \sum_{m=0}^{\infty} (r_{10} r_{12}^2 p^2)^m \]

Summing the geometric series as previously

\[ = r_0 + t_0 t_{10} r_{12} p^2 \frac{1}{1 - r_{10} r_{12} p^2} \]

The individual reflection and transmission coefficients can be determined using the Fresnel equation. Recall

\[ r = \frac{Q - Q'}{Q + Q'}, \quad t = \frac{2Q}{Q + Q'} \]
Fresnel Equation Identity

Applying the Fresnel equations to the top interface
Applying the Fresnel equations to the top interface

\[ r_{01} = \frac{Q_0 - Q_1}{Q_0 + Q_1} \]
Fresnel Equation Identity

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\[ r_{01} = \frac{Q_0 - Q_1}{Q_0 + Q_1} \]

\[ t_{01} = \frac{2Q_0}{Q_0 + Q_1} \]
Applying the Fresnel equations to the top interface

\[ r_{01} = \frac{Q_0 - Q_1}{Q_0 + Q_1} \]  
\[ t_{01} = \frac{2Q_0}{Q_0 + Q_1} \]

\[ r_{10} = \frac{Q_1 - Q_0}{Q_1 + Q_0} = -r_{01} \]
Applying the Fresnel equations to the top interface

\[ r_{01} = \frac{Q_0 - Q_1}{Q_0 + Q_1} \]

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\[ t_{01} = \frac{2Q_0}{Q_0 + Q_1} \]

\[ t_{10} = \frac{2Q_1}{Q_1 + Q_0} \]
Fresnel Equation Identity

Applying the Fresnel equations to the top interface

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we can, therefore, construct the following identity

\[ r_{01}^2 + t_{01} t_{10} \]
Fresnel Equation Identity

Applying the Fresnel equations to the top interface

\[ r_{01} = \frac{Q_0 - Q_1}{Q_0 + Q_1} \quad t_{01} = \frac{2Q_0}{Q_0 + Q_1} \]

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we can, therefore, construct the following identity

\[ r_{01}^2 + t_{01}t_{10} = \frac{(Q_0 - Q_1)^2}{(Q_0 + Q_1)^2} + \frac{2Q_0}{Q_0 + Q_1} \cdot \frac{2Q_1}{Q_1 + Q_0} \]
Fresnel Equation Identity

Applying the Fresnel equations to the top interface

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\begin{align*}
 r_{01} &= \frac{Q_0 - Q_1}{Q_0 + Q_1} \\
 t_{01} &= \frac{2Q_0}{Q_0 + Q_1} \\
 r_{10} &= \frac{Q_1 - Q_0}{Q_1 + Q_0} = -r_{01} \\
 t_{10} &= \frac{2Q_1}{Q_1 + Q_0}
\end{align*}
\]

we can, therefore, construct the following identity

\[
\begin{align*}
 r_{01}^2 + t_{01}t_{10} &= \left(\frac{Q_0 - Q_1}{Q_0 + Q_1}\right)^2 + \frac{2Q_0}{Q_0 + Q_1} \cdot \frac{2Q_1}{Q_1 + Q_0} \\
 &= \frac{Q_0^2 + 2Q_0Q_1 + Q_1^2}{(Q_0 + Q_1)^2}
\end{align*}
\]
Fresnel Equation Identity

Applying the Fresnel equations to the top interface

\[ r_{01} = \frac{Q_0 - Q_1}{Q_0 + Q_1} \quad \quad t_{01} = \frac{2Q_0}{Q_0 + Q_1} \]

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we can, therefore, construct the following identity

\[ r_{01}^2 + t_{01}t_{10} = \frac{(Q_0 - Q_1)^2}{(Q_0 + Q_1)^2} + \frac{2Q_0}{Q_0 + Q_1} \frac{2Q_1}{Q_1 + Q_0} \]

\[ = \frac{Q_0^2 + 2Q_0Q_1 + Q_1^2}{(Q_0 + Q_1)^2} = \frac{(Q_0 + Q_1)^2}{(Q_0 + Q_1)^2} = 1 \]
Reflection Coefficient of a Slab

Starting with the reflection coefficient of the slab obtained earlier

\[
\begin{align*}
    r_{\text{slab}} &= r_{01} + t_{01} t_{10} r_{12} \\
    &= r_{01} + (1 - r_{201}) r_{12} p_2 \\
    &= r_{01} + r_{201} r_{12} p_2 + (1 - r_{201}) r_{12} p_2 - r_{10} r_{12} p_2.
\end{align*}
\]

Using the identity 
\[ t_{01} t_{10} = 1 - r_{201} \]
expanding over a common denominator

and if 
\[ n_0 = n_2 \]
then 
\[
    r_{01} = -r_{12} + r_{01} r_{12} p_2.
\]
Reflection Coefficient of a Slab

Starting with the reflection coefficient of the slab obtained earlier

\[ r_{slab} = r_{01} + t_{01} t_{10} r_{12} p^2 \frac{1}{1 - r_{10} r_{12} p^2} \]
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Starting with the reflection coefficient of the slab obtained earlier

\[ r_{slab} = r_{01} + t_{01} t_{10} r_{12} p^2 \frac{1}{1 - r_{10} r_{12} p^2} \]

\[ = r_{01} + \left(1 - r_{01}^2\right) r_{12} p^2 \frac{1}{1 - r_{10} r_{12} p^2} \]

Using the identity

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\[ = r_{01} + \left(1 - r_{01}^2\right) r_{12} p^2 \frac{1}{1 - r_{10} r_{12} p^2} \]

\[ = \frac{r_{01} + r_{01}^2 r_{12} p^2 + \left(1 - r_{01}^2\right) r_{12} p^2}{1 - r_{10} r_{12} p^2} \]

Using the identity

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Reflection Coefficient of a Slab

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\[ = \frac{r_{01} + r_{01}^2 r_{12} p^2 + \left(1 - r_{01}^2\right) r_{12} p^2}{1 - r_{10} r_{12} p^2} \]

\[ r_{slab} = \frac{r_{01} - r_{12} p^2}{1 + r_{01} r_{12} p^2} \]

Using the identity

\[ t_{01} t_{10} = 1 - r_{01}^2 \]

expanding over a common denominator
Reflection Coefficient of a Slab

Starting with the reflection coefficient of the slab obtained earlier

\[ r_{slab} = r_0 + t_0 t_{10} r_{12} p^2 \frac{1}{1 - r_{10} r_{12} p^2} \]

\[ = r_0 + \left(1 - r_{01}^2\right) r_{12} p^2 \frac{1}{1 - r_{10} r_{12} p^2} \]

\[ = \frac{r_0 + r_{01}^2 r_{12} p^2 + \left(1 - r_{01}^2\right) r_{12} p^2}{1 - r_{10} r_{12} p^2} \]

Using the identity

\[ t_0 t_{10} = 1 - r_{01}^2 \]

expanding over a common denominator

and if \( n_0 = n_2 \) then

\[ r_{01} = -r_{12} \]
Reflection Coefficient of a Slab

Starting with the reflection coefficient of the slab obtained earlier

\[ r_{\text{slab}} = r_{01} + t_{01} t_{10} r_{12} p^2 \frac{1}{1 - r_{10} r_{12} p^2} \]

\[ = r_{01} + (1 - r_{01}^2) r_{12} p^2 \frac{1}{1 - r_{10} r_{12} p^2} \]

\[ = \frac{r_{01} + r_{01}^2 r_{12} p^2 + (1 - r_{01}^2) r_{12} p^2}{1 - r_{10} r_{12} p^2} \]

\[ r_{\text{slab}} = \frac{r_{01} - r_{12} p^2}{1 + r_{01} r_{12} p^2} = \frac{r_{01} (1 - p^2)}{1 - r_{01}^2 p^2} \]

Using the identity

\[ t_{01} t_{10} = 1 - r_{01}^2 \]

expanding over a common denominator

and if \( n_0 = n_2 \) then

\[ r_{01} = -r_{12} \]
Kiessig Fringes

\[ p^2 = e^{iQ_1 \Delta} \]

\[ r_{slab} = \frac{r_{01} (1 - p^2)}{1 - r_{01}^2 p^2} \]

If we plot the reflectivity
\[ R_{slab} = |r_{slab}|^2 \]

There are Kiessig fringes which arise from interference between reflections at the top and bottom of the slab. These have an oscillation frequency
\[ 2\pi/\Delta = 0.0092 \text{Å}^{-1} \]
Kiessig Fringes

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Kiessig Fringes

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There are Kiessig fringes which arise from interference between reflections at the top and bottom of the slab. These have an oscillation frequency

\[ 2\pi / \Delta = 0.0092 \text{Å}^{-1} \]
Kinematical Reflection from a Thin Slab

If the slab is thin, we can make further simplifications.
Kinematical Reflection from a Thin Slab

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Kinematical Reflection from a Thin Slab

If the slab is thin, we can make further simplifications

\[ r_{slab} = \frac{r_{01} (1 - p^2)}{1 - r_{01}^2 p^2} \]

\[ |r_{01}| \ll 1 \quad \alpha > \alpha_c \]
Kinematical Reflection from a Thin Slab

If the slab is thin, we can make further simplifications

\[ r_{slab} = \frac{r_{01} (1 - p^2)}{1 - r_{01}^2 p^2} \approx r_{01} (1 - p^2) \]

\[ |r_{01}| \ll 1 \quad \alpha > \alpha_c \]
Kinematical Reflection from a Thin Slab

If the slab is thin, we can make further simplifications

\[ r_{slab} = \frac{r_{01} (1 - p^2)}{1 - r_{01}^2 p^2} \]

\[ \approx r_{01} (1 - p^2) \approx r_{01} \left(1 - e^{iQ\Delta}\right) \]

\[ |r_{01}| \ll 1 \quad \alpha > \alpha_c \]
Kinematical Reflection from a Thin Slab

If the slab is thin, we can make further simplifications

\[ r_{slab} = \frac{r_{01} (1 - p^2)}{1 - r_{01}^2 p^2} \]

\[ \approx r_{01} (1 - p^2) \]

\[ = r_{01} \left( 1 - e^{iQ\Delta} \right) \]

\[ |r_{01}| \ll 1 \quad \alpha > \alpha_c \]

\[ q_1 \approx q_0 + i b_\mu / q_0 \]

Since \( Q\Delta \ll 1 \) for a thin slab

\[ r_{thinslab} \approx -i \lambda \rho r_0 \Delta \sin \alpha \]
Kinematical Reflection from a Thin Slab

If the slab is thin, we can make further simplifications

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\[ = r_{01} (1 - e^{iQ\Delta}) \]

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\[ q_1 \approx q_0 + ib_\mu / q_0 \]

\[ b_\mu = (2k_\mu) / Q_c^2 \]
Kinematical Reflection from a Thin Slab

If the slab is thin, we can make further simplifications

\[ r_{\text{slab}} = \frac{r_{01}(1 - p^2)}{1 - r_{01}^2 p^2} \]

\approx r_{01} (1 - p^2)

\approx r_{01} \left(1 - e^{iQ\Delta}\right)

| r_{01} | \ll 1 \quad \alpha > \alpha_c

q_1 \approx q_0 + ib_{\mu}/q_0

b_{\mu} = \frac{(2k_{\mu})}{Q_c^2} \sim \frac{Q_c}{Q_c^2} = 1/Q_c
Kinematical Reflection from a Thin Slab

If the slab is thin, we can make further simplifications

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\[ |r_{01}| \ll 1 \quad \alpha > \alpha_c \]

\[ q_1 \approx q_0 + ib_\mu/q_0 \]

\[ b_\mu = (2k_\mu)/Q_c^2 \sim Q_c/Q_c^2 = 1/Q_c \]

\[ r_{01} = \frac{q_0^2 - q_1^2}{(q_0 + q_1)^2} \]
Kinematical Reflection from a Thin Slab

If the slab is thin, we can make further simplifications

\[ r_{slab} = \frac{r_{01} (1 - p^2)}{1 - r_{01}^2 p^2} \]

\[ \approx r_{01} (1 - p^2) \]

\[ = r_{01} \left(1 - e^{iQ\Delta}\right) \]

\[ |r_{01}| \ll 1 \quad \alpha > \alpha_c \]

\[ q_1 \approx q_0 + ib_\mu/q_0 \]

\[ b_\mu = (2k_\mu)/Q_c^2 \sim Q_c/Q_c^2 = 1/Q_c \]

\[ r_{01} = \frac{q_0^2 - q_1^2}{(q_0 + q_1)^2} \approx \frac{b_\mu^2}{q_0^2 (2q_0)^2} \]
Kinematical Reflection from a Thin Slab

If the slab is thin, we can make further simplifications

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since \( Q \Delta \ll 1 \) for a thin slab

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Multilayers in the Kinematical Regime

$N$ repetitions of a bilayer of thickness $\Lambda$ composed of two materials, $A$ and $B$ which have a density contrast ($\rho_A > \rho_B$).
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From a stack of $N$ bilayers

$$r_N(\zeta) = \sum_{\nu=0}^{N-1} r_1(\zeta) e^{i2\pi\zeta\nu} e^{-\beta\nu}$$
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Reflectivity of a Bilayer

The reflectivity from a single bilayer can be evaluated using the reflectivity developed for a slab but replacing the density of the slab material with the difference in densities of the bilayer components.
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The total reflectivity for the multilayer is therefore:

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\[ \beta = 2 \left[ \frac{\mu_A}{2} \frac{\Gamma \Lambda}{\sin \theta} + \frac{\mu_B}{2} \frac{(1 - \Gamma) \Lambda}{\sin \theta} \right] = \frac{\Lambda}{\sin \theta} \left[ \mu_A \Gamma + \mu_B (1 - \Gamma) \right] \]
When $\zeta = Q \Lambda / 2 \pi$ is an integer, we have peaks.

As $N$ becomes larger, these peaks would become more prominent.

This is effectively a diffraction grating for x-rays.

Multilayers are used commonly on laboratory sources as well as at synchrotrons as mirrors.
Reflectivity Calculation

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$$R_{\text{Multilayer}}$$

W/Si multilayer
10 bilayers on Si
$\Delta_{W}\Delta_{Si}=10\text{Å}/40\text{Å}$
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Slab - Multilayer Comparison

\[ \Delta = 68 \text{ Å} \]

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C. Segre (IIT)